Daniel Bernoulli (1700-1782)

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A Unique Dynasty of Scientists

Daniel Bernoulli is one of the prominent members of the Bernoulli family from Basel, Switzerland, whose members excelled in various theoretical and applied scientific fields — especially in mathematics, probability theory, physics and medicine — in the second half of the seventeenth and in the eighteenth centuries. The family originated from Antwerp, once under the domination of Catholic Spain. It emigrated in 1567 to Frankfurt, Germany, because of its Calvinist faith, and in the end settled in Basel in 1620. Until Niklaus Bernoulli (1623–1708), the important wealth of the family came from the spice trade: Niklaus was himself a merchant and an officer of the city of Basel. But three of his sons, Jakob (1754–1705), Nikolaus (1662–1716) and Johann (1767–1748), took another route. Nikolaus was a painter and a member of the Municipality of Basel. Jakob studied philosophy and theology, and Johann medicine, but they both became

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renowned mathematicians, developing in particular differential and integral calculus and siding with Gottfried Wilhelm Leibniz in his quarrel with Isaac Newton — the phrase "integral calculus" is due to Jakob.

Daniel Bernoulli, was born in Groningen on 8 February 1700, where his father had taught at the university since 1695, and died in Basel on 17 March 1782. He was Johann's son and the cousin of Nikolaus (1687–1759) — alias Nikolaus I, just as his uncles were named Jakob I and Johann I by historians to distinguish them from the younger members of this dynasty, who had the same Christian names (see Figure 1) — a son of Nikolaus the painter. Daniel was a doctor in medicine and his cousin in jurisprudence but, like their predecessors, their main achievements were in mathematics and the sciences. Also like their predecessors, they travelled in Europe and were part of a network of the most eminent scientists of the time, and members of many scientific academies. (On the main members of the Bernoulli family, see O'Connor and Robertson 1997–98).

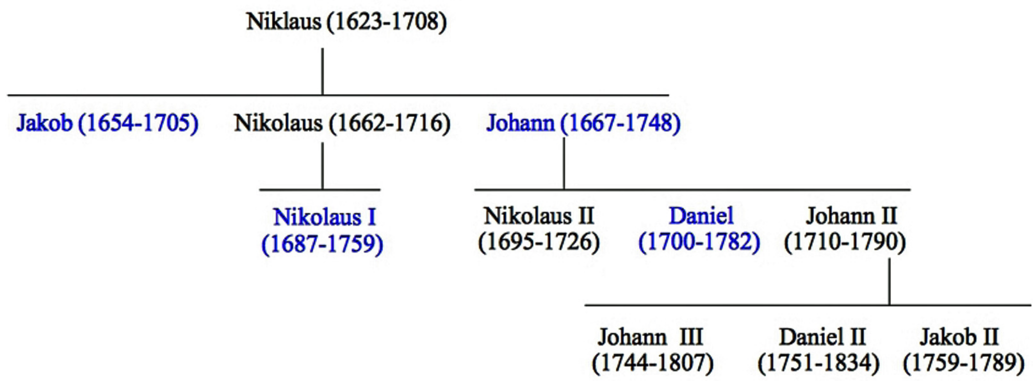

Figure 1. The main members of the Bernoulli family.

As regards the "moral sciences", the works of Jakob, Nikolaus and Daniel are of outstanding interest thanks to their path-breaking contributions to probability theory (Hacking 1975; Daston 1988; Hald 1990). Jakob is the author of the celebrated *Ars Conjectandi*, mainly written between 1684 and 1689 but posthumously published in 1713. It was the first book written on the theory of probability, this "art of conjecturing" that he also proposed to call "stochastics". (Pierre Rémond de Montmort's 1708 *Essay d'analyse des jeux de hasard*, the second edition of which was also published in 1713, was in fact written a long time after Jakob Bernoulli drafted his manuscript.) Jakob's book elaborated considerably on Christiaan Huygens's "De ratiociniis in ludo aleæ", a paper included as an appendix in Frans van Schooten's 1657 book, *Exercitationum mathematicarum*. It deals, *inter alia*, with the specification of the concept of (mathematical) expectation proposed by Blaise Pascal and Huygens, the statement and proof of the binomial distribution and the (weak) "law of large numbers" (as Siméon-Denis Poisson called it later). Moreover, it introduced a first clear distinction between objective (frequentist or statistical) probabilities, where the frequencies of events are calculated from experiments or observations, and subjective (or epistemic) probabilities due to our imperfect knowledge and measuring the degree of our belief, or our "reason to believe" (Condorcet), in a statement or a proposition about things or events (see Hald 1990: 28–9, 245–7; and Daston 1988: ch. 4, 1994, for the evolution of these concepts in the eighteenth and nineteenth centuries). In addition, we owe Jakob Bernoulli also "the important distinction between probabilities which can be calculated a priori (deductively, from considerations of symmetry) and those which can be calculated only a posteriori (inductively, from relative frequencies)" (Hald 1990: 247). Finally, the fourth and unfinished part of *Ars Conjectandi*, which contains the statement and proof of the law of large numbers, is entitled "Usum & applicationem præcedentis doctrinæ in civilibus, moralibus & œconomicis" ("The use and application of the previous doctrine to civil, moral and economic affairs"). It posed the fundamental question of the use of the mathematical developments of "expectatio", made for the games of chance, to the more traditional field of the "probability" of judgements developed for example in jurisprudence — that is, in modern parlance, the application of probability theory to social and economic matters. Jakob could not bring his project to an end but considered it as "the main part" of his work (to Leibniz, 3 October

1703, in JEHPS 2006: 5). This part was to inspire Nikolaus's approach and, some decades later, Condorcet's research programme.

Nikolaus who, contrary to the legend, was not the editor of Jakob's book (Kohli 1975; Yushkevich 1987), continued the work of his uncle. In his 1709 thesis, *Dissertatio inauguralis mathematico-juridica de usu artis conjectandi in jure*, and in "Specimina artis conjectandis, ad quaestiones juris applicatae" — an abridged version of his thesis, published in 1711 in a supplement of the Leibnizian *Acta Eruditorum* — he used probability theory to deal with juridical and economic questions such as the reliability of witnesses and of suspicions, marine insurance, the probability of human life, life annuities, or the problem of the "absent" (after how many years can an absent person be considered as dead?) (see, for example, Hald 1990: ch. 21).

Daniel Bernoulli and Moral Sciences

From Basel to Saint Petersburg to Basel

Daniel Bernoulli's father, Johann, wanted his son to become a merchant, but Daniel was more attracted by mathematics. As a compromise he studied medicine but, just as his uncle Jakob and his father Johann, wanted to embark on an academic career. He could not immediately obtain a position in Basel. He travelled in Italy and published his first book in 1724, *Exercitationes Quaedam Mathematicae* (*Mathematical Exercises*), a collection of his essays edited with the help of his friend Christian Goldbach, which contains some developments in applied mathematics and physics. It is in this field that he was remarkably inventive, and his fundamental book on hydrodynamics, *Hydrodynamica*, written as early as 1733 but only published in 1738, is still remembered as a milestone in the discipline. From 1725 onwards, Daniel won several scientific prizes, ten of them awarded by the Paris Académie des Sciences. In 1725 he reluctantly accepted a position in the Saint Petersburg Imperial Academy of Sciences, newly established at the instigation of Leibniz.

While happy to work there with Leonhard Euler (a fellow countryman from Basel) who arrived in 1727, he was nevertheless relieved to leave in 1733 and, in 1734, started botany lectures in Basel, then switched to physiology in 1743 and finally to physics, and held this latter chair from 1750 to 1776.

As regards the moral sciences, Daniel's best known achievements concern the possible applications of probability theory to individual and collective decision-making: first with his celebrated essay, "Specimen Theoriae Novae de Mensura Sortis" ("Proposal of a new theory of the measure of chance") — presented in 1731 in Saint Petersburg and published there in 1738 in the *Annals* of the academy — and, second, with his intervention in the controversy over the desirability or not for the public authority to recommend the inoculation of smallpox to fight against this malady (Bernoulli 1760 [1766]). The theoretical context was a long-standing discussion over the nature and applicability of the calculus of probability, and especially of a central concept: (mathematical) expectation. Many detailed objections were levelled against this calculus, in particular by one of the most celebrated mathematicians and philosophers of the age, Jean Le Rond d'Alembert (Daston 1979; Paty 1988), who also intervened in the controversy over inoculation and reacted to Bernoulli's memoir (Daston 1988: 82–9, Paty 1988: 220–25). The Saint Petersburg paper, which will be dealt with here, belongs to an early stage of this controversy. It forms a milestone in utility theory, the apprehension of risk and the formalisation of economic theory. It is also the symbol of a transformation in the notion of rationality and the definition of the conduct of the "reasonable" man: from a question of justice and equity in jurisprudence and games of chance (with, in particular, aleatory contracts and fair stakes) symbolized by the figure of the disinterested judge abstracting from personal and subjective situations, this conduct changed into that of the prudent man, the model of whom is the merchant in a risky environment (Daston 1980, 1989). Daniel Bernoulli stressed this shift of points of view when, in his essay, he referred to a letter Nikolaus sent him in 1732:

> [H]e [Nikolaus] declared that he was in no way dissatisfied with my proposition on the evaluation of risky propositions when applied to the case of a man who is to evaluate his own prospects. However, he thinks that the case is different if a third person, somewhat in the position of a judge, is to evaluate the prospects of any participant in a game in accord with equity and justice. (Bernoulli 1738 [1954]: 33)

How it all started: Nikolaus's challenge and the first discussions

The long-standing debate on what was called later the "Saint Petersburg paradox" (see, for example, Samuelson 1977; Jorland 1987; Daston 1988: ch. 2; Dutka 1988; Martin 2014) started with a letter of Nikolaus to Montmort, dated 9 September 1713 and immediately published by Montmort, with some other correspondence with Johann and Nikolaus, in the fifth part of the second edition of his *Essay* (Rémond de Montmort 1713: 401–2). Nikolaus proposed Montmort five problems to solve, the last two being the following:

> *Fourth problem.* A promises B to give him one *écu* if, with a normal dice, he obtains six points at the first roll, two *écus* if he succeeds at the second roll, three *écus* if he succeeds at the third roll, four *écus* at the fourth roll, and so on; one asks, what is B's expectation. *Fifth problem.* The same thing is asked if A promises B to give him écus in this progression: 1, 2, 4, 8, 16, etc. or 1, 3, 9, 27, etc. ... instead of 1, 2, 3, 4, 5, etc. like before. (ibid.: 402)

Montmort answered (ibid.: 407) that the solution, based on the calculus of the limit of infinite series developed by Nikolaus's uncle Jakob, was easy to find. In a subsequent correspondence however (Dutka 1988: 19; Meusnier 2006: 9–11), Nikolaus pointed out to Montmort two difficulties that the latter had disregarded. Two methods could be used to solve the problem: B's expected gain could be found either as the sum of the terms of an infinite series, as stressed by his correspondent, or using mathematical induction. But while the solution to the fourth problem poses no problem (B's expectation is 6) whatever the method employed, two important discrepancies arise instead in the case of the fifth problem — discrepancies which shake the belief in the meaning and relevance of (mathematical) expectation. In the first place, in the case of the first progression for example — $1, 2, 2^2, 2^3, \ldots 2^n \ldots$ — and because the probability to obtain a "six" at every roll is one-sixth, the first method gives a solution:

$$\frac{1}{6} \,.\, 1 + \frac{1}{6}\left(\frac{5}{6}\right) .\, 2 + \frac{1}{6}\left(\frac{5}{6}\right)^2 .\, 2^2 + \ldots + \frac{1}{6}\left(\frac{5}{6}\right)^n .\, 2^n + \ldots = \frac{1}{6}\sum_{n=0}^{\infty}\left(\frac{5}{6}\right)^n .\, 2^n$$

which is infinite, while, with the second method, the result is different: it is finite and moreover negative: $-1/4$ (Dutka 1988: 19; Meusnier 2006: 10–11). In the second place, since in games of chance the fair stake was defined as the player's expected gain, another discrepancy arises with what "good sense" would advise: no reasonable player is supposed to pay an infinitely large sum of money — or even only an important sum, compared to his wealth — to play this game.

The first difficulty was very embarrassing and forms the real paradox in this story. After Gabriel Cramer rekindled the debate in 1728 (Montmort had died in 1719), the discussion concentrated on the second difficulty — which was not really a paradox, in spite of the name given to it, but a discrepancy between theory and reality. It involved Cramer, in correspondence with Nikolaus and Georges-Louis Leclerc de Buffon, and Nikolaus with Daniel. An excerpt of this correspondence between Nikolaus and Cramer is quoted at the end of Daniel's essay (Bernoulli 1738 [1954]: 33–5), and the exchange of letters between Buffon and Cramer is recalled in Buffon's "Essai d'arithmétique morale" (Leclerc de Buffon 1777: 75–7). The game considered is now the toss of a coin, with the same progression of possible gains $(1, 2, 2^2, 2^3, \ldots 2^n \ldots)$. The game goes on if B obtains "tails" and stops with "heads". With the first method — the probability to obtain "heads" at every toss being one-half — B's expectation of gain is again infinite:

$$\frac{1}{2}.1+\frac{1}{2^2}.2+\frac{1}{2^3}.2^2+\ldots=\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+1}2^n$$

The debate thus turned around the possibility to change the definition of expectation because the usual one gave results at odds with "good sense": this was possible by changing the apprehension of the possible gains. As Cramer put it in 1728 — an approach which turned out to be the same as Buffon's and Daniel's solutions — the reason of the discrepancy between the mathematical calculation and common sense "results from the fact that, in theory, mathematicians evaluate money in proportion to its quantity while, in practice, people with common sense evaluate money in proportion to the use they can make of it" (Cramer, in Bernoulli 1738 [1954]: 33, translation modified). For the player, what matters is not the sum of money, but, in Daniel's words, the "emolumentum" — the benefit, the advantage, usually translated as "utility" — an individual gains from it, which depends on the wealth already possessed. Daniel formalized and developed this idea in an outstanding way.

The Book of Daniel

A given sum of money does not have the same importance or utility to different persons with different wealth. The greater is an individual's wealth ("summa bonorum"), the less a given increment of it will be of importance to its owner. If, with Daniel Bernoulli, infinitesimal increments of wealth are considered, this means that what is called today the marginal utility of wealth is decreasing. This is what had become to be known as "Bernoulli's hypothesis". It is moreover to be noted that Bernoulli's concept of wealth is defined in a broad and modern way: it does not only consist in the material wealth already possessed, but it takes also into account the future incomes that a given human capital is susceptible to yield (Bernoulli 1738 [1954]: 25). Daniel illustrated his approach with Figure 2 — which in modern parlance represents the utility of wealth — where the horizontal axis denotes wealth and its possible increments (not infinitesimal here for the sake of clarity) and the vertical axis the benefit or utility obtained from them. The function is concave because of the positive but decreasing marginal utility. AB denotes the initial wealth before the game starts. By convention, the utility of AB is nil, so that the utilities CG, DH, EL and FM of the possible increments of wealth BC, BD, BE, BF, can be read easily on the figure.

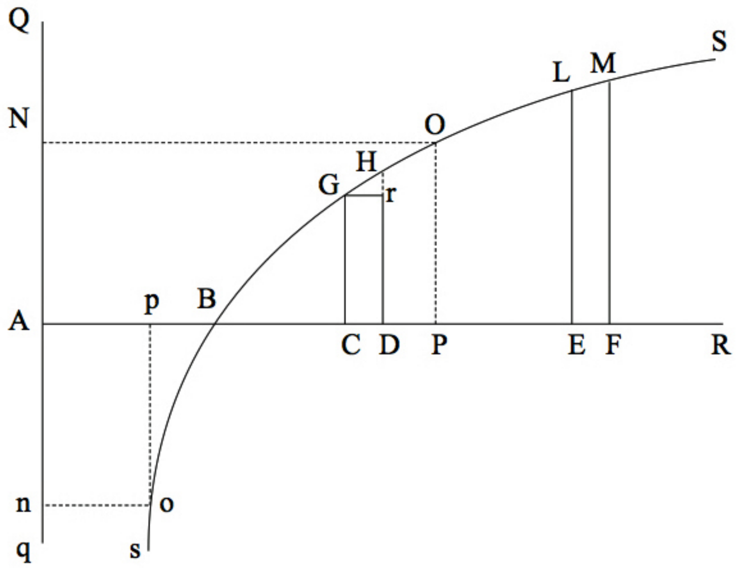

Figure 2. Daniel Bernoulli's representation of the utility of wealth.

In order to be more precise and to apply his approach to concrete cases (marine insurance, for example), Daniel advanced a second hypothesis: the increment of utility generated by an infinitesimal increase of wealth is inversely proportional to this wealth (Bernoulli 1738 [1954]: 25). In modern parlance, this is to suppose that the utility function is iso-elastic, the algebraic value of the elasticity of the marginal utility of wealth with respect to wealth being moreover equal to -1.

The (utility) function, Bernoulli writes, can thus be specified. Let α denote the initial wealth AB. At any point of the curve (for example C), if x denotes the wealth, dx its increment, y the utility of x, and dy its increment (the segment rH in the graph), then, b being a positive constant:

$$dy = b\left(\frac{1}{x}\right)dx \quad \Rightarrow \quad y = b\log x + k$$

where k is a constant of integration. Since, by convention, as stated above, $y = 0$ for $x = \alpha$:

$$k = -b\log\alpha \quad \Rightarrow \quad y = b\log\frac{x}{\alpha}$$

Since in a game of chance the mathematical expectation of the monetary gains can no longer be considered as the fair stake a player has to pay, what has to be calculated is instead the "emolumentum medium", that is the "mean utility" or, as Cramer put it in 1728, the "espérance morale" (moral expectation) of the player — Condorcet later spoke of "espérance relative" (relative expectation). The formula of the moral expectation is simply that of the mathematical expectation, the possible gains BC, BD, BE, BF, etc. having simply to be replaced with their respective utility CG, DH, EL and FM, and so on. Suppose that there are m independent ways of obtaining BC, n of obtaining BD, p for BE, q for BF, and so on. Then, for the player, the "moral expectation" of the gains — PO in Figure 2 — is given by:

$$PO = \frac{mCG + nDH + pEL + qFM + \ldots}{m + n + p + q + \ldots}$$

with the segment BP on the horizontal axis denoting the corresponding (and finite) expected gain.

Now, to get into the game, a player will never pay a sum, the disutility of which is greater than the moral expectation of the gain. As Cramer himself put it in 1728, the stake should be "of such a magnitude that the pain caused by its loss is equal to the moral expectation of the pleasure I hope to derive from my gain" (in Bernoulli 1738 [1954]: 34). How much, then, will a player be ready to pay? Let $po = PO$ be the disutility of the maximum stake. It is easy to see in Figure 2 that this maximum stake is Bp — that is, the diminution of the initial wealth of the player, the disutility of which is precisely equal to po. Owing to the concavity of the curve, $Bp < BP$, that is, the greatest stake that the player should be prepared to pay is inferior to the expected gain. The values of all the variables can be calculated. In the case of the game of heads or tails mentioned above, the stake would be of a few *écus* only.

This also shows that, whenever the stake is determined on the basis of the usual rule based on the expected gain, the player will always be a loser because the disutility of the stake would always be greater than the utility of the expected gain. However, the two rules of the mathematical and moral expectations are (1) equivalent if utility is directly proportional to the gain, in which case the function becomes a straight line, or (2) nearly equivalent in case the initial wealth is "infinitely great" compared to the greatest possible gain, in which case the graph of the function is approximately a straight line (Corollaries I and II, in Bernoulli 1738 [1954]: 27).

Daniel applied his new method of evaluating risk to questions of trade and insurance (Bernoulli 1738 [1954]: 29–30), for example to state the conditions of profitable insurance, both for the merchant who thinks about insuring his trade and the insurer who insures the merchant. He showed also that the merchant can reduce his risk in dividing his merchandise and sending it on several boats instead of one single ship (ibid.: 30–31) — "it is advisable to divide goods which are exposed to some danger into several portions rather than to risk them all together" (ibid.: 30). The analysis of a better risk spread can also be extended to other questions. "This counsel will be equally serviceable for those who invest their fortunes in foreign bills of exchange and other hazardous enterprises" (ibid.: 31).

Another important theme can also be found in Daniel's 1738 paper: the definition of risk aversion as a situation in which a player prefers a gain which is certain to a greater but uncertain (expected) gain — a case illustrated nowadays with a concave utility function. As he wrote at the beginning of his text, still referring to the criterion of mathematical expectation:

> Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. (Bernoulli [1738] 1954: 24, s. 3)

Bernoulli's presentation of moral expectation was later taken up and developed in the expected utility theory, and his iso-elastic utility function is still widely used in problems of applied microeconomics. To conclude, two remarks are in order as regards the shape of the utility function. On the one hand, even when Bernoulli's contemporaries accepted his first hypothesis (the decreasing marginal utility of wealth), the second, concerning the value of the elasticity of the marginal utility of wealth with respect to wealth, was contested. Condorcet, in particular, supposed that the absolute value of the elasticity was greater than one — with important consequences in favour of progressive taxation (Faccarello 2006: 26–30). On the other hand, the logarithmic form of the utility function is not the only one contemplated by Daniel Bernoulli. Cramer, in the 1728 letter to Nikolaus, extensively quoted by Daniel at the end of his essay, proposed $y = \sqrt{x}$. It is worth noting that Daniel accepted this specification as a possible solution. The manner in which Cramer expressed "the basic principle . . . that reasonable men should evaluate money in proportion to the use they can make thereof", he wrote, is "in perfect agreement with our view" (in Bernoulli 1738 [1954]: 34, translation modified).

See also:

Marie-Jean-Antoine-Nicolas Caritat de Condorcet; Formalization and Mathematical Modelling; French Enlightenment; Information and Uncertainty.

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